Notes for Programs

1 Introduction

This document describes the programs used in "Emerging Market Business Cycles: The Cycle is the Trend." The Matlab programs described below include:

1. **solve_uhlig.m**: Solves the linear system.
2. **calculate_moments.m**: Calculates key theoretical moments given a set of parameters.
3. **key_moments.m**: A function that takes a vector of parameters as inputs and generates ten moments as outputs.

Some notes on the linearization of the model is appended to this document. See the paper or the appended notes for an explanation of the model and the corresponding notation.

2 Solution of the Linear System

The solution of the linear system is done by **solve_uhlig.m**, which in turn calls Harald Uhlig's toolkit (available from *www.wiwi.hu-berlin.de/wpol/html/toolkit.htm*). We first solve the linear system, and then extend the matrices to handle non-normalized variables. A superscript "0" denotes the normalized problem. Define $\mathbf{x}_t^0 = (\hat{k}_{t+1}, \hat{b}_{t+1})'$ as our endogenous state variables. Note the timing convention that $(\hat{k}_{t+1}, \hat{b}_{t+1})$ are chosen at time t. The exogenous state variables are $\mathbf{z}_t = (g_t, z_t)'$, which follow

$$\mathbf{z}_t = NN\mathbf{z}_{t-1} + \begin{pmatrix} \epsilon_t^g \\ \epsilon_t^z \end{pmatrix}$$

where

$$NN = \begin{pmatrix} \rho_g & 0 \\ 0 & \rho_z \end{pmatrix}$$

and

$$E\left((\epsilon^g, \epsilon^z)'(\epsilon^g, \epsilon^z)'\right) = \Sigma = \begin{pmatrix} \sigma_g^2 & 0 \\ 0 & \sigma_z^2 \end{pmatrix}.$$

Our endogenous "jump" variables are: $\mathbf{y}_t^0 = (\hat{c}_t, l_t, n_t, \hat{i}_t, \hat{y}_t, q_t, nx_t)'$. These are chosen conditional on $\mathbf{x}_{t-1}$ and $\mathbf{z}_t$. The Uhlig toolkit solves for PP^0, QQ^0, RR^0, SS^0 such that

$$\mathbf{x}_t^0 = PP^0\mathbf{x}_{t-1}^0 + QQ^0\mathbf{z}_t \tag{1}$$

$$\mathbf{y}_t^0 = RR^0\mathbf{x}_{t-1}^0 + SS^0\mathbf{z}_t \tag{2}$$

After solving the normalized system, we extend the endogenous state space to include G_{t-1} and the endogenous choice variables to include the non-normalized levels of consumption, investment and income: $\mathbf{x}_t = (\hat{k}_{t+1}, \hat{b}_{t+1}, G_t) = (\mathbf{x}_t^0, G_t)'$ and $\mathbf{y}_t = (\hat{c}_t, l_t, n_t, \hat{i}_t, \hat{y}_t, q_t, nx_t, c_t, i_t, y_t)'$ $= (\mathbf{y}_t^0, c_t, i_t, y_t)'$. The fact that $G_t = G_{t-1} + g_t$ implies:

$$PP = \begin{bmatrix} PP^0 & \mathbf{0}_{(n_{PP^0}\times 1)} \\ \mathbf{0}_{(1\times m_{PP^0})} & 1 \end{bmatrix} \tag{3}$$

and

$$QQ = \begin{bmatrix} QQ^0 \\ 1 \quad 0 \end{bmatrix}, \tag{4}$$

where m_A and n_A are the number of rows and columns, respectively, of any matrix A.

RR is extended according to the definition $a_t = \hat{a}_t + G_{t-1}$, for $a = c, i, y$:

$$RR = \begin{bmatrix} RR^0 & \mathbf{0}_{(n_{RR^0}\times 1)} \\ & \tilde{RR} \end{bmatrix} \tag{5}$$

where rows corresponding to c, i, y of $\tilde{RR}$ are $[RR^0(m,:), 1]$, where m denotes the respective row of $\hat{c}_t, \hat{i}_t, \hat{y}_t$ in RR^0. Similarly for SS:

$$SS = \begin{bmatrix} SS^0 \\ \tilde{SS} \end{bmatrix}, \tag{6}$$

where $\tilde{SS} = [SS^0(m,:)]$, for $m = \hat{c}_t, \hat{i}_t, \hat{y}_t$.

3 Calculating Theoretical Moments

The program **calculate_moments.m** calculates the theoretical moments of the linearized model. We follow Burnside (1999) fairly closely. The moments calculated are (where a tilde denotes HP filtered variables):

$$\left(\sigma(\tilde{y}), \sigma(\Delta y), \frac{\sigma(\tilde{i})}{\sigma(\tilde{y})}, \frac{\sigma(\tilde{c})}{\sigma(\tilde{y})}, \frac{\sigma(\tilde{nx})}{\sigma(\tilde{y})}, \rho(\tilde{y}_t, \tilde{y}_{t-1}), \rho(\Delta y_t, \Delta y_{t-1}), \rho(\tilde{c}, \tilde{y}), \rho(\tilde{i}, \tilde{y}), \rho(\tilde{n}, \tilde{y})\right)'.$$

Define $\mathbf{s}_t = (\mathbf{x}^0_{t-1}, \mathbf{z}_t) = (\hat{k}_t, \hat{b}_t, g_t, z_t)'$ and

$$M = \begin{bmatrix} PP^0 & QQ^0 \\ \mathbf{0} & NN \end{bmatrix}$$

so that

$$\mathbf{s}_t = M\mathbf{s}_{t-1} + \epsilon_t$$

where $\epsilon_t = (0, 0, \epsilon^g_t, \epsilon^z_t)'$. Let $\tilde{\Sigma} = E(\epsilon\epsilon') = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Sigma \end{bmatrix}$.

Define $F = (\hat{c}, n, \hat{i}, nx, \hat{y})$, and let H_1 and H_2 be the corresponding rows from RR and SS so that

$$F_t = H_1\mathbf{x}_{t-1} + H_2\mathbf{z}_t.$$

Let $\Gamma_0 \equiv E(\mathbf{s}\mathbf{s}')$ denote the variance-covariance matrix of $\mathbf{s}$:

$$\Gamma_0 = M\Gamma_0 M' + \tilde{\Sigma}.$$

This can be solved as

$$vec(\Gamma_0) = (I - (M \otimes M))^{-1}\, vec(\tilde{\Sigma}).$$

The k^{th} autocovariance can be obtained by $\Gamma_k = E(\mathbf{s}_t\mathbf{s}'_{t-k}) = M^k\Gamma_0$.

We now calculate the variance-covariance matrix of the first difference of our endogenous variables. Define

$$\Delta F_t \equiv \begin{pmatrix} \hat{c}_t - \hat{c}_{t-1} + g_{t-1} \\ n_t - n_{t-1} \\ \hat{i}_t - \hat{i}_{t-1} + g_{t-1} \\ nx_t - nx_{t-1} \\ \hat{y}_t - \hat{y}_{t-1} + g_{t-1} \end{pmatrix} = \begin{pmatrix} c_t - c_{t-1} \\ n_t - n_{t-1} \\ i_t - i_{t-1} \\ nx_t - nx_{t-1} \\ y_t - y_{t-1} \end{pmatrix}.$$

Note that this is not simply the first difference of F, but adds g_{t-1} for c, i, and y. Next define

$$\bar{\mathbf{s}}_t = \begin{pmatrix} \mathbf{x}^0_{t-1} - \mathbf{x}^0_{t-2} + \Theta_1 \mathbf{z}_{t-1} \\ \mathbf{z}_t \\ \mathbf{z}_{t-1} \end{pmatrix},$$

where

$$\Theta_1 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

That is, $\Theta_1 \mathbf{z}_{t-1} = \begin{pmatrix} g_{t-1} \\ g_{t-1} \end{pmatrix}$. This implies

$$\bar{\mathbf{s}}_t = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & I \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{s}_t + \begin{pmatrix} -I & \Theta_1 \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} \mathbf{s}_{t-1} \tag{7}$$

$$= \mathbf{B}_0 \mathbf{s}_t + \mathbf{B}_1 \mathbf{s}_{t-1} \tag{8}$$

Defining $\bar{\Gamma}_0 = E(\bar{\mathbf{s}}\bar{\mathbf{s}}')$, we have

$$\bar{\Gamma}_0 = \mathbf{B}_0 \Gamma_0 \mathbf{B}_0' + \mathbf{B}_0 \Gamma_1 \mathbf{B}_1' + \mathbf{B}_1 \Gamma_0' \mathbf{B}_0' + \mathbf{B}_1 \Gamma_0 \mathbf{B}_1'.$$

To compute the variance-covariance of ΔF, note that $\mathbf{x}^0_t = PP^0 \mathbf{x}^0_{t-1} + QQ^0 \mathbf{z}_t$ implies

$$\begin{aligned} \mathbf{x}^0_t - \mathbf{x}^0_{t-1} + \Theta_1 \mathbf{z}_t &= PP^0(\mathbf{x}^0_{t-1} - \mathbf{x}^0_{t-2}) + QQ^0(\mathbf{z}_t - \mathbf{z}_{t-1}) + \Theta_1 \mathbf{z}_t && (9) \\ &= PP^0(\mathbf{x}^0_{t-1} - \mathbf{x}^0_{t-2} + \Theta_1 \mathbf{z}_{t-1})) + (QQ^0 + \Theta_1)\mathbf{z}_t - (QQ^0 + PP^0\Theta_1)\mathbf{z}_{t-1} \\ &= \left[PP^0, QQ^0 + \Theta_1, -(QQ^0 + PP^0\Theta_1)\right] \bar{\mathbf{s}}_t && (10) \end{aligned}$$

We can then write

$$\bar{\mathbf{s}}_{t+1} = \bar{M}\bar{\mathbf{s}}_t + \bar{\epsilon}_{t+1},$$

where

$$\bar{M} = \begin{pmatrix} PP^0, & QQ^0 + \Theta_1, & -(QQ^0 + PP^0\Theta_1) \\ \mathbf{0} & NN & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{pmatrix}$$

and

$$\bar{\epsilon}_{t+1} = \begin{pmatrix} \mathbf{0} \\ \epsilon^g_{t+1} \\ \epsilon^z_{t+1} \\ \mathbf{0} \end{pmatrix}.$$

We now define a matrix that pulls out g_{t-1} to add to $\Delta\hat{c}_t$, $\Delta\hat{i}_t$, and $\Delta\hat{y}_t$:

$$\Theta_f \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We have

$$\begin{aligned} \Delta F_t &= F_t - F_{t-1} + \Theta_f \mathbf{z}_{t-1} && (11) \\ &= H_1(\mathbf{x}^0_{t-1} - \mathbf{x}^0_{t-2}) + H_2(\mathbf{z}_t - \mathbf{z}_{t-1}) + \Theta_f \mathbf{z}_{t-1} && (12) \\ &= H_1(\mathbf{x}^0_{t-1} - \mathbf{x}^0_{t-2} + \Theta_1 \mathbf{z}_{t-1}) + H_2 \mathbf{z}_t + (\Theta_f - H_1\Theta_1 - H_2)\, \mathbf{z}_{t-1} \\ &= \begin{bmatrix} H_1, & H_2, & \Theta_f - H_1\Theta_1 - H_2 \end{bmatrix} \bar{s}_t \\ &= \bar{H}\bar{\mathbf{s}}_t && (13) \end{aligned}$$

Therefore, $Var_0(\Delta F) = E(\Delta F \Delta F')$ can be computed as

$$Var(\Delta F) = \bar{H}\bar{\Gamma}_0\bar{H}'.$$

The autocovariance of ΔF can be computed as

$$
\begin{aligned}
Var_k(\Delta F) \equiv E(\Delta F \Delta F'_{-k}) &= \bar{H} E(\bar{\mathbf{s}}_t \bar{\mathbf{s}}'_{t-k}) \bar{H}' \\
&= \bar{H} \bar{\Gamma}_k \bar{H}' \\
&= \bar{H} \bar{M}^k \bar{\Gamma}_0 \bar{H}'
\end{aligned} \tag{14}
$$

4 HP Filtered Moments

The HP filtered series can be obtained from the original series using the two-sided filter: $F^{HP} = B(L)F$, where $B(L) = \sum_{j=-\infty}^{\infty} b_j L^j$.[1] When the HP parameter λ is set to 1600, the coefficients b_j are given by (following Burnside, who follows King and Rebelo) $b_j = b_{-j} = -(0.894^j)(0.0561\cos(.112j) + 0.0558\sin(.112j))$. For $j = 0$, $b_0 = 1 - 0.0561 = 0.9439$.

To obtain F^{HP}, we start from the first-differenced series. Note that for any series $F^{HP} = B(L)F = B(L)(1-L)^{-1}\Delta F$. Define $\tilde{B}(L) = B(L)(1-L)^{-1}$. This implies that $(1-L)\tilde{B}(L) = B(L)$. We then have

$$
\begin{aligned}
(1-L)\tilde{B}(L) &= \sum_j \tilde{b}_j L^j - \sum_j \tilde{b}_j L^{j+1} \\
&= \sum_j (\tilde{b}_j - \tilde{b}_{j-1}) L^j
\end{aligned} \tag{15}
$$

Equating terms we have

$$\tilde{b}_j - \tilde{b}_{j-1} = b_j.$$

Or, $\tilde{b}_j = \sum_{s=-\infty}^{j} b_s$. Therefore $F^{HP} = \sum_{j=-\infty}^{\infty} \tilde{b}_j \Delta F$. (In practice, we define $\tilde{b}_j = \sum_{s=-2N}^{j} b_j$ for $N = 500$.)

[1] An alternative is to calculate the moments using the frequency domain. We found the computations were faster in the time domain.

To get the moments of F^{HP}, we use

$$\begin{aligned}
Var_i(F^{HP}) = E(F_t^{HP} F_{t-i}^{HP'}) &= E(\{\tilde{B}(L)\Delta F\}\{\tilde{B}(L)L^i \Delta F'\}) \\
&= \sum_{j=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} \tilde{b}_j \tilde{b}_{j'} E(L^j \Delta F L^{j'+i} \Delta F') \\
&= \sum_{j=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} \tilde{b}_j \tilde{b}_{j'} Var_{j-j'-i}(\Delta F)
\end{aligned}$$

Defining $k = j' + i - j$ and using $Var_{-k}(\Delta F) = Var_k(\Delta F)'$:

$$\begin{aligned}
Var_i(F^{HP}) &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \tilde{b}_j \tilde{b}_{j-k-i} Var_k(\Delta F)' \\
&= Var_0(\Delta F) \sum_{j=-\infty}^{\infty} \tilde{b}_j \tilde{b}_{j-i} + \sum_{k=1}^{\infty} \sum_{j=-\infty}^{\infty} \left(\tilde{b}_j \tilde{b}_{j+k-i} Var_k(\Delta F) + \tilde{b}_j \tilde{b}_{j-k-i} Var_k(\Delta F)' \right)
\end{aligned}$$

Using $Var_i(\Delta F) = \bar{H}\bar{\Gamma}_k \bar{H}'$, we have

$$Var_i(F^{HP}) = \bar{H} \left(\bar{\Gamma}_0 \sum_{j=-\infty}^{\infty} \tilde{b}_j \tilde{b}_{j-i} + \sum_{k=1}^{\infty} \left(\bar{\Gamma}_k \sum_{j=-\infty}^{\infty} \tilde{b}_j \tilde{b}_{j+k-i} + \bar{\Gamma}_k' \sum_{j=-\infty}^{\infty} \tilde{b}_j \tilde{b}_{j-k-i} \right) \right) \bar{H}'$$

To implement this in practice, we choose $N = 500$ and $M_n = 100$ and compute

$$\begin{aligned}
Var_i(F^{HP}) \approx\ & \bar{H}(\bar{\Gamma}_0 \sum_{j=-N+i}^{N} \tilde{b}_j \tilde{b}_{j-i} \\
+ & \sum_{k=1}^{i} \bar{M}^k \bar{\Gamma}_0 \sum_{j=-N+i-k}^{N} \tilde{b}_j \tilde{b}_{j-i+k} \\
+ & \sum_{k=1}^{i} \bar{\Gamma}_0 \bar{M}^{k'} \sum_{j=-N+i+k}^{N} \tilde{b}_j \tilde{b}_{j-i-k} \\
+ & \sum_{k=i+1}^{M_n} \bar{M}^k \bar{\Gamma}_0 \sum_{j=-N-i+k}^{N} \tilde{b}_j \tilde{b}_{j+i-k} \\
+ & \sum_{k=i+1}^{M_n} \bar{\Gamma}_0 \bar{M}^{k'} \sum_{j=-N+i+k}^{N} \tilde{b}_j \tilde{b}_{j-i-k}) \bar{H}'
\end{aligned} \tag{16}$$

Log-linearization Notes

1 Problem

$$V = \max \frac{\left(C^{\gamma}\mathcal{L}^{1-\gamma}\right)^{1-\sigma}}{1-\sigma} + \beta G^{\gamma(1-\sigma)} EV(K', B', Z', G') \tag{1}$$

subject to

$$C + GK' = Y(K, N, z, G) + (1-\delta)K - \frac{\phi}{2}\left(G\frac{K'}{K} - \mu_g\right)^2 K - B + GQB' \tag{2}$$

where

$$\begin{aligned} \ln(Z') &\equiv z' = \rho_z z + \epsilon^z \\ \ln(G') &\equiv g' = (1-\rho_g)\ln(\mu_g) + \rho_g g + \epsilon^g \end{aligned} \tag{3}$$

and

$$Y = ZK^{1-\alpha}(GN)^{\alpha} \tag{4}$$

$$Q^{-1} = 1 + r^* + \psi\left(e^{B'-\bar{B}} - 1\right) \tag{5}$$

2 First Order Conditions

$$\gamma C^{\gamma(1-\sigma)-1}\mathcal{L}^{(1-\gamma)(1-\sigma)}\left(G\phi\left(G\frac{K'}{K} - \mu_g\right) - G\right) = \beta G^{\gamma(1-\sigma)} V_{K'} \tag{6}$$

$$\gamma C^{\gamma(1-\sigma)-1}\mathcal{L}^{(1-\gamma)(1-\sigma)} GQ = -\beta G^{\gamma(1-\sigma)} EV_{B'} \tag{7}$$

$$\frac{C}{\mathcal{L}} = \frac{\gamma}{1-\gamma} Y_N \tag{8}$$

3 Envelope Conditions

$$V_{K'} = \gamma C'^{\gamma(1-\sigma)-1} \mathcal{L}'^{(1-\gamma)(1-\sigma)} \left(Y_{K'} + 1 - \delta + \frac{\phi}{2} \left(\left(g' \frac{K''}{K'} \right)^2 - \mu_g^2 \right) \right) \tag{9}$$

$$V_{B'} = -\gamma C'^{\gamma(1-\sigma)-1} \mathcal{L}'^{(1-\gamma)(1-\sigma)} \tag{10}$$

4 Steady State Relationships

$$\begin{aligned}
\bar{Q} &= \frac{1}{1+r^*} = \beta \mu_g^{\gamma(1-\sigma)-1} && (11) \\
(1-\alpha)\frac{\bar{Y}}{\bar{K}} &= \frac{1}{\bar{Q}} - 1 + \delta \\
\frac{\bar{C}}{\bar{Y}} &= 1 + (1-\delta-\mu_g)\frac{\bar{K}}{\bar{Y}} + (\mu_g \bar{Q} - 1)\frac{\bar{B}}{\bar{Y}} \\
\bar{N} &= \left(1 + \frac{\bar{C}}{\bar{Y}} \frac{1-\gamma}{\alpha\gamma}\right)^{-1} \\
\bar{K} &= \left(\mu_g \frac{\bar{K}}{\bar{Y}}\right)^{\frac{1}{\alpha}} \bar{N} \\
\bar{Y} &= \frac{\bar{Y}}{\bar{K}} \bar{K} \\
\bar{C} &= \frac{\bar{C}}{\bar{Y}} \bar{Y} \\
\bar{X} &= (\mu_g - 1 + \delta)\, \bar{K} \\
\bar{nx} &= \frac{\bar{Y} - \bar{C} - \bar{X}}{\bar{Y}}
\end{aligned}$$

5 Log-linearized Model

5.1 K' FOC:

$$\gamma C^{\gamma(1-\sigma)-1}\mathcal{L}^{(1-\gamma)(1-\sigma)}G\left(\phi\left(G\frac{K'}{K}-\mu_g\right)+1\right)\approx \tag{12}$$
$$\gamma\mu_g\bar{C}^{\gamma(1-\sigma)-1}\bar{\mathcal{L}}^{(1-\gamma)(1-\sigma)}\left((\gamma(1-\sigma)-1)\hat{c}+(1-\gamma)(1-\sigma)\hat{l}+(1+\phi\mu_g)\hat{g}+\phi\mu_g(\hat{k}'-\hat{k})\right)$$

$$\begin{aligned}
& \beta G^{\gamma(1-\sigma)}\gamma C'^{\gamma(1-\sigma)-1}\mathcal{L}'^{(1-\gamma)(1-\sigma)}\left(Y_{K'}+1-\delta+\frac{\phi}{2}\left(\left(g'\frac{K''}{K'}\right)^2-\mu_g^2\right)\right)\approx \\
& \gamma\beta\mu_g^{\gamma(1-\sigma)}\bar{C}^{\gamma(1-\sigma)-1}\bar{\mathcal{L}}^{(1-\gamma)(1-\sigma)} * \\
& \{(1-\delta+(1-\alpha)\frac{\bar{Y}}{\bar{K}})(\gamma(1-\sigma)\hat{g}+(\gamma(1-\sigma)-1)\hat{c}'+(1-\gamma)(1-\sigma)\hat{l}') \\
+ \quad & (1-\alpha)\frac{\bar{Y}}{\bar{K}}(\hat{y}'-\hat{k}') \\
+ \quad & \phi\mu_g^2(\hat{g}'+\hat{k}''-\hat{k}')\}
\end{aligned} \tag{13}$$

Equating (12) to the expectation of (13) and using $\beta\mu_g^{\gamma(1-\sigma)-1}=\left(1-\delta+(1-\alpha)\frac{\bar{Y}}{\bar{K}}\right)^{-1}$:

$$\begin{aligned}
0 \quad = \quad & (\gamma(1-\sigma)-1)E\hat{c}' \\
+ \quad & (1-\gamma)(1-\sigma)E\hat{l}' \\
+ \quad & \beta\mu_g^{\gamma(1-\sigma)+1}\phi E\hat{g}' \\
+ \quad & \beta\mu_g^{\gamma(1-\sigma)-1}(1-\alpha)\frac{\bar{Y}}{\bar{K}}E\hat{y}' \\
+ \quad & \beta\mu_g^{\gamma(1-\sigma)+1}\phi E\hat{k}'' \\
- \quad & \left(\beta\mu_g^{\gamma(1-\sigma)-1}\left((1-\alpha)\frac{\bar{Y}}{\bar{K}}+\phi\mu_g^2\right)+\phi\mu_g\right)\hat{k}' \\
- \quad & (\gamma(1-\sigma)-1)\hat{c} \\
- \quad & (1-\gamma)(1-\sigma)\hat{l} \\
+ \quad & (\gamma(1-\sigma)-1-\phi\mu_g)\,\hat{g} \\
+ \quad & \phi\mu_g\hat{k}
\end{aligned} \tag{14}$$

5.2 B' FOC

$$
\begin{aligned}
&\gamma C^{\gamma(1-\sigma)-1}\mathcal{L}^{(1-\gamma)(1-\sigma)}GQ \approx \\
&\gamma \bar{C}^{\gamma(1-\sigma)-1}\bar{\mathcal{L}}^{(1-\gamma)(1-\sigma)}\mu_g \bar{Q} * \\
&\Big((\gamma(1-\sigma)-1)\hat{c} + (1-\gamma)(1-\sigma)\hat{l} + \hat{g} + \hat{q}\Big)
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&\beta G^{\gamma(1-\sigma)}\gamma C'^{\gamma(1-\sigma)-1}\mathcal{L}'^{(1-\gamma)(1-\sigma)} \approx \\
&\beta\gamma\mu_g^{\gamma(1-\sigma)}\bar{C}^{(\gamma(1-\sigma)-1)}\bar{\mathcal{L}}^{(1-\gamma)(1-\sigma)} * \\
&\Big(\gamma(1-\sigma)\hat{g} + (\gamma(1-\sigma)-1)\hat{c'} + (1-\gamma)(1-\sigma)\hat{l'}\Big)
\end{aligned} \tag{16}
$$

Equating (15) to the expectation of (16) and using $\bar{Q} = \beta\mu_g^{\gamma(1-\sigma)-1}$:

$$
\begin{aligned}
0 &= (\gamma(1-\sigma)-1)E\hat{c'} && (17)\\
&+ (1-\gamma)(1-\sigma)E\hat{l'} \\
&+ (\gamma(1-\sigma)-1)\hat{g} && (18)\\
&- (\gamma(1-\sigma)-1)\hat{c} \\
&- (1-\gamma)(1-\sigma)\hat{l} \\
&- \hat{q}
\end{aligned}
$$

5.3 Labor-Leisure

$$
\begin{aligned}
\frac{C}{\mathcal{L}} &= \frac{\gamma}{1-\gamma}\alpha\frac{Y}{N} \\
&\approx> \\
0 &= \hat{y} - \hat{n} - \hat{c} + \hat{l}
\end{aligned} \tag{19}
$$

If $\gamma = 1$, then $\hat{l} = 0$.

5.4 Budget Constraint

$$
\begin{aligned}
0 &= Y + GQB' - B - X - C && (20)\\
&\approx \bar{Y}\hat{y} + \mu_g \bar{Q}\bar{B}(\hat{b}' + \hat{g} + \hat{q}) - \bar{B}\hat{b} - \bar{X}\hat{x} - \bar{C}\hat{c}
\end{aligned}
$$

where investment $X \equiv GK' - (1-\delta) - \frac{\phi}{2}\left(G\frac{K'}{K} - \mu_g\right)^2 K$. Or

$$
\bar{X}\hat{x} \approx \bar{K}\left(\mu_g \hat{k}' - (1-\delta)\hat{k} + \mu_g \hat{g}\right) \tag{21}
$$

5.5 Technology

$$
\begin{aligned}
Y &= ZK^{1-\alpha}(GN)^{\alpha} && (22)\\
\hat{y} &\approx \hat{z} + (1-\alpha)\hat{k} + \alpha(\hat{g} + \hat{n})
\end{aligned}
$$

5.6 Leisure = 1-Labor

$$
\begin{aligned}
\mathcal{L} &= 1 - N && (23)\\
\bar{\mathcal{L}}\hat{l} &\approx -\bar{N}\hat{n}
\end{aligned}
$$

Note that if $\gamma = 1$, then $\bar{N} = 1$ and $\bar{\mathcal{L}} = 0$, and this reduces to $\hat{n} = 0$.

5.7 Interest Rate Function

$$
\begin{aligned}
Q^{-1} &= 1 + r^* + \psi\left(e^{B' - \bar{B}} - 1\right) && (24)\\
\hat{q} &\approx -\psi \bar{B}\bar{Q}\hat{b}'
\end{aligned}
$$

5.8 Net Exports – *Linearized*

$$
\begin{aligned}
nx &\equiv \frac{NX}{Y} = 1 - \frac{X}{Y} - \frac{C}{Y} \qquad (25)\\
\Delta nx &\approx (1 - \bar{nx})\hat{y} - \frac{\bar{X}}{\bar{Y}}\hat{x} - \frac{\bar{C}}{\bar{Y}}\hat{c}
\end{aligned}
$$